Dirac operator on the Riemann sphere

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Abstract

We solve for spectrum, obtain explicitly and study group properties of eigenfunctions of Dirac operator on the Riemann sphere S^2. The eigenvalues λ are nonzero integers. The eigenfunctions are two-component spinors that belong to representations of $SU(2)$-group with half-integer angular momenta $l = |\lambda| - \frac{1}{2}$. They form on the sphere a complete orthonormal functional set alternative to conventional spherical spinors. The difference and relationship between the spherical spinors in question and the standard ones are explained.

PACS: 02.20.Qs, 02.30.Gp, 02.30.Nw, 02.30.Tb.

KEYWORDS: Dirac operator, spherical functions, spherical spinors, orthogonal polynomials, monopole harmonics.

arXiv: hep-th/0212134

Introduction

The question of spectrum and eigenstates of Dirac operator on the Riemann sphere S^2 is a well posed mathematical problem that may be solved exactly. The resulting family of orthogonal two-component spinors plays the role of orthogonal spherical functions in problems with half-integer momenta. Therefore the solutions are of fundamental value.

The eigenspinors were first found by Newman and Penrose [1] and the relation with SU(2) matrix elements was deduced later in [2]. These eigenspinors also reappear from time to time under the name of "monopole harmonics", see, for instance [3]. A general construction of Dirac operator eigenfunctions on N-dimensional spheres was given in paper [4] and the two-dimensional case was specially addressed in [5] (see also [6]).

One must mention several physical situations where the solutions may find application. The first involves the so-called spectral boundary conditions. Those have been introduced by Atiah, Patodi and Singer [7] and are widely used now in studies of spectral properties of operators in the limited space [8]. Being compatible with chiral symmetry these boundary conditions are perfectly suited for investigations of chirality breaking in quantum field theory.

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The spectral boundary conditions are defined in terms of eigenvalues of Dirac operator on the boundary. Thus our problem bears a direct relation to the (spherical) spectral bag that allows to hold quarks while preserving the chiral symmetry.

Another application refers to physics of electrons in substances called fullerenes. Their molecules consist of carbon atoms located in vertices of polyhedra or on the surface of clusters with a spheroidal crystallographic structure. The most abundant and the most spherical is the C_{60} molecule (known also as Buckminsterfullerene or buckyball) that looks like a standard soccer ball. The interest to electronic structure of fullerenes was heated by their unconventional superconducting properties with transition temperatures higher than those of the classic superconductors (reaching up to 60 K!). In the continuum limit the conducting electrons of fullerene molecules obey the Dirac equation [9] and our results are pertinent for the continuous description of electron states in fullerenes.

Our present goal was to implement an explicit classical approach to eigenstates of the Dirac operator with the intention to make them a tool that could be later employed in physical problems. A short presentation of the topic was given in [10]. In principle one could extract the required information from the already cited papers but this would take some effort.

It was shown in [4] that for Dirac operator on the N-dimensional sphere one may separate the dependence on principal polar angle and solve the obtained second order differential equation. The result was written in terms of Jacobi polynomials. This led to a recurrent formula that related spinor spherical functions on S^N to those on S^{N-1}. The subsequent reduction (quite conceivable for S^2 though) was left to the reader.

The two-dimensional case was considered as an example in [5] (Chapter 9) and may be also found in [6]. However there the problem was elegantly solved with the help of complex coordinates on the Riemann sphere whereas the transformation to standard spherical coordinates was left aside.

With practical applications in mind we shall try to make the paper self-contained. It has the following structure. Section 1 introduces the notation and expounds on general properties of Dirac operator on the Riemann sphere. The spectrum and eigenfunctions themselves are investigated in Section 2. We start from the eigenvalues (Sect. 2.1), analyse $SU(2)$ properties of solutions (Sect. 2.2), then write them out explicitly (Sect. 2.3) and check their properties with respect to time-reversal (Sect. 2.4). Section 3 explains the link between the newly found spherical spinors and the conventional ones [11]. After reminding the spinor transformation rules (Sect. 3.2) we convert our eigenspinors to Cartesian coordinates and derive the relationship in question (Sect 3.3). The conclusion is followed by two technical Appendices dwelling on properties of Jacobi polynomials (Appendix A) and spherical functions (Appendix B).

1 Generalities

We shall start with introducing spherical coordinates and writing down the Dirac operator for free massless fermions on the Riemann sphere S^2. Then we shall show that Dirac operator has has no zero eigenvalues on the sphere.

The sphere of unit radius S^2 may be parameterized by two spherical angles $q^1 = \theta$, $q^2 = \phi$ that are related to Cartesian coordinates x, y, z as follows:

$$x = \sin\theta\cos\phi; \quad y = \sin\theta\sin\phi; \quad z = \cos\theta. \tag{1}$$

The metric tensor and the natural diagonal zweibein on the sphere are well known:

$$g_{\alpha\beta} = \mathrm{diag}(1,\, \sin^2\theta); \qquad e^a_\alpha = \mathrm{diag}(1,\, \sin\theta); \quad g_{\alpha\beta} = \delta_{ab}\, e^a_\alpha\, e^b_\beta. \tag{2}$$

From here on we shall denote zweibein and coordinate indices by Latin and Greek letters respectively. Multiplication by the zweibein converts the two types of indices into each other:

$$A_a = e^\alpha_a\, A_\alpha; \qquad A^a = e^a_\alpha\, A^\alpha; \qquad e^a_\alpha\, e^\alpha_b = \delta^a_b. \tag{3}$$

In curvilinear coordinates the ordinary Euclidean partial derivatives must be replaced by covariant ones. For quantities with Latin indices those include the spin connection $R^a_{\alpha\, b}$[1]:

$$D_\alpha A^a = \partial_\alpha A^a + R^a_{\alpha\, b}\, A^b \qquad \text{and} \qquad D_\alpha A_a = \partial_\alpha A_a - A_b\, R^b_{\alpha\, a}. \tag{4}$$

In our case the nonzero components of spin connection are:

$$R^{12}_\phi = -R^{21}_\phi = -\cos\theta. \tag{5}$$

Spinors in two dimensions have two components and the role of Dirac matrices belongs to Pauli matrices: $\gamma^a \longrightarrow (\sigma_x,\, \sigma_y)$ where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Covariant derivatives of 2-spinors are also expressed in terms of the spin connection:

$$\nabla_\alpha \psi = \partial_\alpha \psi + \frac{i}{4} R^{ab}_\alpha\, \sigma_{ab}\, \psi. \tag{7}$$

Here σ_{ab} are the rotation generators for spin-$\frac{1}{2}$ fields:

$$\sigma_{1\,2} = -\sigma_{2\,1} = -\frac{i}{2}[\gamma_1,\, \gamma_2] = \sigma_z. \tag{8}$$

A standard calculation gives for covariant derivatives:

$$\nabla_\theta = \partial_\theta \qquad \text{and} \qquad \nabla_\phi = \partial_\phi - \frac{i\sigma_z}{2} \cos\theta. \tag{9}$$

Now we may define the Dirac operator. In spherical coordinates it is given by the convolution of covariant derivatives in spinor representation with the zweibein and σ-matrices:

$$-i\hat{\nabla} = -i\, e^{\alpha\, a} \sigma_a \nabla_\alpha. \tag{10}$$

With the help of definitions (2, 6, 9) it is straightforward to obtain:

$$-i\,\hat{\nabla} = -i\sigma_x \left(\partial_\theta + \frac{\cot\theta}{2} \right) - \frac{i\sigma_y}{\sin\theta}\, \partial_\phi. \tag{11}$$

Finally let us show that Dirac operator on the sphere has no zero eigenvalues. This is a general property of manifolds with positive curvature known as Lichnerowicz theorem [12].

[1] Since there is actually no difference between upper and lower Latin indices the second of eqs. (4) is nothing but the skew symmetry of $\hat{R}$.

Consider the square of Dirac operator $(-i\hat{\nabla})^2$. Obviously if $-i\hat{\nabla}$ had a zero eigenvalue then $(-i\hat{\nabla})^2$ must have it either. Let us split the product of σ-matrices in $(\hat{\nabla})^2$ into symmetric and antisymmetric parts $\sigma^\alpha\sigma^\beta = \frac{1}{2}\{\sigma^\alpha,\, \sigma^\beta\} + \frac{1}{2}[\sigma^\alpha,\, \sigma^\beta]$. Implementing the relation between commutators of covariant derivatives and curvature we obtain:

$$(-i\hat{\nabla})^2 + \nabla_F^2 = -\frac{i}{2}\sigma^{\alpha\beta}\,[\nabla_\alpha,\, \nabla_\beta] = \frac{1}{4}\, R^{\alpha\beta}_{\alpha\beta}. \tag{12}$$

Here $R^{\alpha\beta}_{\alpha\beta}$ is the trace of Riemann curvature tensor and ∇_F^2 is the covariant Laplace operator in fundamental representation of the $SU(2)$-group:

$$\nabla_F^2 = g^{\alpha\beta}\,(\nabla_\alpha\nabla_\beta - \Gamma^\gamma_{\alpha\beta}\nabla_\gamma) = \frac{1}{\sqrt{g}}\,\nabla_\alpha\, g^{\alpha\beta}\sqrt{g}\,\nabla_\beta, \tag{13}$$

where $\nabla_{\alpha,\,\beta,\,\gamma}$ are covariant derivatives in fundamental representation (7), $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol and g is the determinant of metric, $g = \det\|g_{\alpha\beta}\| = \sin^2\theta$.

The result (12) is called the Lichnerowicz formula. In the case of sphere $R^{\alpha\beta}_{\alpha\beta} = 2$ so that

$$(-i\hat{\nabla})^2 + \nabla^2 = \frac{1}{2}. \tag{14}$$

(certainly this may be proved directly). The positive difference is brought about by the curvature of Riemann sphere. The covariant Laplacian is a non-positive operator $\nabla^2 \leq 0$. Hence, by virtue of (14), $(-i\hat{\nabla})^2$ must be strictly positive and Dirac operator $-i\hat{\nabla}$ has no zero eigenvalues.

In conclusion I would like to present the explicit formulae for the square of Dirac operator and the fundamental Laplace operator on the unit sphere:

$$(-i\hat{\nabla})^2 = -\frac{1}{\sin\theta}\partial_\theta\,\sin\theta\,\partial_\theta - \frac{\partial_\phi^2}{\sin^2\theta} + i\sigma_z\frac{\cos\theta}{\sin^2\theta}\partial_\phi + \frac{1}{4} + \frac{1}{\sin^2\theta}; \tag{15a}$$

$$-\nabla_F^2 = -\frac{1}{\sin\theta}\partial_\theta\,\sin\theta\,\partial_\theta - \frac{\partial_\phi^2}{\sin^2\theta} + i\sigma_z\frac{\cos\theta}{\sin^2\theta}\partial_\phi - \frac{1}{4} + \frac{1}{\sin^2\theta}. \tag{15b}$$

Obviously the difference between the two agrees with (14).

2 Dirac operator on the sphere

Now we shall study the eigenvalue problem for Dirac operator on the sphere S^2. First we shall derive the differential equation for spinor components, find eigenvalues and the general form of solutions. Then we shall study properties of the eigenfunctions that arise from the $SU(2)$-invariance of the problem and classify the solutions. Finally we shall present them explicitly and discuss their behaviour under the complex conjugation and time-reversal.

2.1 The eigenvalue problem

Eigenfunctions of the Dirac operator (11) are two-component spinors that satisfy the eigenvalue equation:

$$-i\,\hat{\nabla}\begin{pmatrix}\alpha_\lambda(\theta,\,\phi)\\ \beta_\lambda(\theta,\,\phi)\end{pmatrix} = \lambda\begin{pmatrix}\alpha_\lambda(\theta,\,\phi)\\ \beta_\lambda(\theta,\,\phi)\end{pmatrix}. \tag{16}$$

This system of first order partial differential equations in α and β allows separation of variables. The first thing is to isolate the ϕ-dependence by expanding the spinors into Fourier series:

$$\left(\begin{array}{c} \alpha_\lambda(\theta,\,\phi) \\ \beta_\lambda(\theta,\,\phi) \end{array} \right) = \sum_m \frac{\exp i\,m\,\phi}{\sqrt{2\pi}} \left(\begin{array}{c} \alpha_{\lambda m}(\theta) \\ \beta_{\lambda m}(\theta) \end{array} \right); \qquad m = \pm\frac{1}{2},\, \pm\frac{3}{2} \ldots \tag{17}$$

where m are half-integers since we work with the spin-$\frac{1}{2}$ field. This converts Eqn. (16) into

$$-i\left(\partial_\theta + \frac{\cot\theta}{2} + \frac{m}{\sin\theta}\right)\beta_{\lambda m}(\theta) \;=\; \lambda\,\alpha_{\lambda m}(\theta); \tag{18a}$$

$$-i\left(\partial_\theta + \frac{\cot\theta}{2} - \frac{m}{\sin\theta}\right)\alpha_{\lambda m}(\theta) \;=\; \lambda\,\beta_{\lambda m}(\theta). \tag{18b}$$

By analogy with the ordinary quantum mechanics the number m may be called the projection of angular momentum onto the polar axis.

Separate equations for spinor components α and β may be obtained by taking the square of Dirac operator (15a):

$$\left[-\frac{1}{\sin\theta}\,\partial_\theta\,\sin\theta\,\partial_\theta + \frac{m^2}{\sin^2\theta} - \sigma_z\frac{m\,\cos\theta}{\sin^2\theta} + \frac{1}{4} + \frac{1}{4\sin^2\theta}\right]\left(\begin{array}{c} \alpha_{\lambda m} \\ \beta_{\lambda m} \end{array} \right) = \lambda^2 \left(\begin{array}{c} \alpha_{\lambda m} \\ \beta_{\lambda m} \end{array} \right). \tag{19}$$

Mark that because of the σ_z present in the third term the equations for upper and lower components are different. As may be seen from eqn. (14) the difference arises already at the level of Laplace operator in fundamental representation (15b).

The further simplification comes from the change of variables $x = \cos\theta$, $x \in [-1,\,1]$, that converts (19) into the generalized hypergeometric equation:

$$\left[\frac{d}{dx}\,(1-x^2)\,\frac{d}{dx} - \frac{m^2 - \sigma_z\,m\,x + \frac{1}{4}}{1-x^2}\right]\left(\begin{array}{c} \alpha_{\lambda m}(x) \\ \beta_{\lambda m}(x) \end{array} \right) = -\left(\lambda^2 - \frac{1}{4}\right)\left(\begin{array}{c} \alpha_{\lambda m}(x) \\ \beta_{\lambda m}(x) \end{array} \right). \tag{20}$$

Take the notice that the replacement $x \to -x$ (or $m \to -m$) is equivalent to trading α for β. Thus the upper and lower spinor components are conjugate with respect to mirror reflection.

Equation (20) is singular at the poles of the sphere $x = \pm 1$. After the redefinition of the unknowns

$$\left(\begin{array}{c} \alpha_{\lambda m}(x) \\ \beta_{\lambda m}(x) \end{array} \right) = \left(\begin{array}{c} (1-x)^{\frac{1}{2}|m-\frac{1}{2}|}(1+x)^{\frac{1}{2}|m+\frac{1}{2}|}\,\xi_{\lambda m}(x) \\ (1-x)^{\frac{1}{2}|m+\frac{1}{2}|}(1+x)^{\frac{1}{2}|m-\frac{1}{2}|}\,\eta_{\lambda m}(x) \end{array} \right), \tag{21}$$

we arrive to the separate equations of hypergeometric type in $\xi_{\lambda m}$ and $\eta_{\lambda m}$:

$$\left\{(1-x^2)\frac{d^2}{dx^2} + \left[\frac{m}{|m|}\sigma_z - (2|m|+2)\,x\right]\frac{d}{dx} - m\,(m+1) + \left(\lambda^2 - \frac{1}{4}\right)\right\}\left(\begin{array}{c} \xi_{\lambda m} \\ \eta_{\lambda m} \end{array} \right) = 0. \tag{22}$$

In order that these equations had on the interval $x \in [-1,\,1]$ square integrable solutions λ must fulfill the condition (see (A.1)–(A.3))

$$\lambda^2 = \left(n + |m| + \frac{1}{2}\right)^2, \tag{23}$$

with non-negative integer $n \geq 0$. Thus λ are nonzero integers

$$\lambda = \pm 1,\, \pm 2 \ldots, \tag{24}$$

and indeed the Dirac operator has no zero-modes. Equations (22) are Jacobi-type equations and their solutions are Jacobi polynomials of the n-th order [13, 14]:

$$\begin{pmatrix} \xi_{\lambda m}(x) \\ \eta_{\lambda m}(x) \end{pmatrix} = \begin{pmatrix} C_\alpha^{mn}\, P_n^{(|m-\frac{1}{2}|,\, |m+\frac{1}{2}|)}(x) \\ C_\beta^{mn}\, P_n^{(|m+\frac{1}{2}|,\, |m-\frac{1}{2}|)}(x) \end{pmatrix}. \tag{25}$$

A brief review of Jacobi polynomials may be found in Appendix A. Properties of solutions (25) will be discussed in Section 2.3.

The constants C_α^{mn}, C_β^{mn} may be fixed by substituting the solutions into equation (16) and normalizing the obtained functions. Let us first find the relation between C_α and C_β. The Dirac operator is sensitive to the sign of m. Applying it to functions (21) at $m > 0$ gives for the Fourier components (the same can be done for $m < 0$):

$$-i\hat{\nabla} \begin{pmatrix} \alpha_{\lambda m} \\ \beta_{\lambda m} \end{pmatrix} = i(1-x^2)^{\frac{m}{2}} \begin{pmatrix} -\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} \left[\left(m+\frac{1}{2}\right)\eta_{\lambda m} - (1-x)\frac{d}{dx}\eta_{\lambda m}\right] \\ \left(\frac{1-x}{1+x}\right)^{\frac{1}{4}} \left[\left(m+\frac{1}{2}\right)\xi_{\lambda m} + (1+x)\frac{d}{dx}\xi_{\lambda m}\right] \end{pmatrix} \tag{26}$$

(Note that coefficients $C_{\alpha,\beta}$ were absorbed in functions ξ, η.) Substituting this into the eigenvalue equation (16) written in terms of ξ and η and using identities (A.7) of the Appendix we find the condition on C_α^{mn}, C_β^{mn} which, after generalization to negative m, takes the form:

$$i\left(n + |m| + \frac{1}{2}\right) \begin{pmatrix} C_\alpha^{mn} \\ -C_\beta^{mn} \end{pmatrix} = \lambda\, \mathrm{sgn}\, m \begin{pmatrix} C_\beta^{mn} \\ C_\alpha^{mn} \end{pmatrix}. \tag{27}$$

Provided that λ is given by (23) this equation has nonzero solutions and the ratio of constants C_α/C_β is defined by the sign of the product $m\lambda$:

$$C_\beta^{mn} = i\, C_\alpha^{mn}\, \mathrm{sgn}(m\,\lambda). \tag{28}$$

The absolute values of the constants can be found from the normalization conditions,

$$\int_0^{2\pi} d\phi \int_0^{\pi} \left[|\alpha_\lambda(\theta,\, \phi)|^2 + |\beta_\lambda(\theta,\, \phi)|^2\right] \sin\theta\, d\theta = 1. \tag{29}$$

This immediately refers us to the norms of Jacobi polynomials with the result:

$$|C_\alpha^{mn}| = |C_\beta^{mn}| = \frac{\sqrt{n!\,(n+2m)!}}{2^{m+\frac{1}{2}}\,\Gamma(n+m+\frac{1}{2})}. \tag{30}$$

The last condition determines the eigenfunctions up to a complex phase that will be fixed in Section 2.4. Presently we shall not write the solutions explicitly leaving this till Section 2.3.

The last remark due here concerns an apparent contradiction between the integer spectrum (24) and formula (14). The latter implies that spectra of the two operators (15) are similar but to the shift $\frac{1}{2}$. Hence according to formula (24) the lowest eigenvalue of the (minus) covariant Laplace operator $-\nabla_F^2$ is $\frac{1}{2}$ and not zero as one could naively expect by analogy with scalar fields. The confusion is readily resolved if one recalls that the aforementioned integer spectrum was obtained for spinors. Unlike scalars those satisfy antiperiodic boundary conditions $\alpha,\,\beta(0) = -\alpha,\,\beta(2\pi)$ and are expanded in half-integer Fourier harmonics. Therefore although formally $-\nabla^2$ and $-\hat{\nabla}^2$ differ only by the numerical constant in fact both spectra and eigenfunctions for scalars and spinors are different.

2.2 The $SU(2)$ algebra

Let us show that the Dirac operator on the sphere S^2 is invariant under transformations of the $SU(2)$ group. The Weyl —Cartan set of generators looks as follows:

$$\hat{L}_z = -i\frac{\partial}{\partial\phi}; \tag{31a}$$

$$\hat{L}_+ = e^{\ i\phi}\left(\frac{\partial}{\partial\theta} + i\,\cot\theta\frac{\partial}{\partial\phi} + \frac{\sigma_z}{2\,\sin\theta}\right); \tag{31b}$$

$$\hat{L}_- = -e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\,\cot\theta\frac{\partial}{\partial\phi} - \frac{\sigma_z}{2\,\sin\theta}\right). \tag{31c}$$

These operators satisfy the standard commutation relations of the $SU(2)$ algebra:

$$\left[\hat{L}_z,\,\hat{L}_+\right] = \hat{L}_+; \tag{32a}$$

$$\left[\hat{L}_z,\,\hat{L}_-\right] = -\hat{L}_-; \tag{32b}$$

$$\left[\hat{L}_+,\,\hat{L}_-\right] = 2\hat{L}_z. \tag{32c}$$

In our case representations of $SU(2)$-group are characterized by half-integer highest weight l (see (37). It will be shown in Section 3.3 that it corresponds to the total angular momentum of the fermion state. The Casimir operator $\hat{L}^2$ takes the value

$$\langle l \mid \hat{L}^2 \mid l\rangle = \langle l \mid \hat{L}_z^2 + \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) \mid l\rangle = l(l+1). \tag{33}$$

The basis vectors of the representation are classified by half-integer projection of angular momentum onto the polar axis that varies in the range $m = l_z = -l, \dots\, l$. The number of basis vector in the representation is $2l+1$. The raising and lowering operators $\hat{L}_+$ and $\hat{L}_-$ change the projection by unity $\hat{L}_\pm \mid l,\,m\rangle \propto \mid l,\,m\pm 1\rangle$.

A direct check proves that generators (31) commute with spherical Dirac operator (11):

$$\left[-i\hat{\nabla},\,\hat{L}_z\right] = \left[-i\hat{\nabla},\,\hat{L}_+\right] = \left[-i\hat{\nabla},\,\hat{L}_-\right] = 0. \tag{34}$$

This means that the Dirac operator is $SU(2)$ invariant and its eigenfunctions may be classified according to their $SU(2)$ transformation properties. Moreover, the action of raising and lowering generators $\hat{L}_+$ and $\hat{L}_-$ transforms its eigenfunctions into the eigenfunctions with the same λ but greater or smaller m respectively.

The square of Dirac operator (15a) and Casimir operator (33) may be diagonalized simultaneously

$$-\hat{\nabla}^2 = \hat{L}^2 + \frac{1}{4}, \tag{35}$$

and their eigenvalues are interrelated:

$$\langle\lambda,\,n|\hat{L}^2|\lambda,\,n\rangle = l(l+1) = \lambda^2 - \frac{1}{4} = (n+|m|)(n+|m|+1). \tag{36}$$

We conclude that the proper values of angular momentum are indeed half-integers related to the eigenvalues of Dirac operator

$$l = |\lambda| - \frac{1}{2} = n + |m| = \frac{1}{2},\,\frac{3}{2},\,\frac{5}{2}\ldots \tag{37}$$

The final remark concerns reducibility of our representation. It is well known that spinor representations in even dimensions are reducible and one may separate the left and right spinor components. One may see that formally this is the case. Operators (31) are diagonal, so the upper and lower spinor components behave with respect to the $SU(2)$-group like two singlets with equal momenta l. However diagonalizing the Dirac operator requires joining them into the pair of doublets with $\lambda = \pm(l+\frac{1}{2})$.

2.3 The spinor spherical functions

Here we shall show how the formerly obtained functions (25) can be grouped into $SU(2)$ multiplets. The multiplets can be constructed in the standard manner, *i. e* by successive application of the lowering operator $\hat{L}_-$ to the highest weight vector, namely to the function with the maximum projection $m = l$ of angular momentum. Let us pass right to the result.

It is quite natural to use instead of n the angular momentum $l = n + |m|$. Let us introduce the integers $l^\pm = l \pm \frac{1}{2}$ and $m^\pm = m \pm \frac{1}{2}$. Having in mind that according to formula (27) solutions for positive and negative λ are different we may write (here again $x = \cos\theta$):

$$\begin{aligned}\Upsilon^\pm_{lm}(x,\,\phi) &= \pm i^{l^+}\,(-1)^{\frac{1}{2}(m^-+|m^-|)}\,\frac{\sqrt{(l+m)!(l-m)!}}{2^{|m|+\frac{1}{2}}\,\Gamma(l^+)} \\ &\times\ \frac{e^{im\phi}}{\sqrt{2\pi}}\begin{pmatrix}\sqrt{\mp i\,\rho^{(|m^-|,\,|m^+|)}(x)}\,P^{(|m^-|,\,|m^+|)}_{l-|m|}(x) \\ \operatorname{sgn} m\,\sqrt{\pm i\,\rho^{(|m^+|,\,|m^-|)}(x)}\,P^{(|m^+|,\,|m^-|)}_{l-|m|}(x)\end{pmatrix};\end{aligned} \tag{38}$$

Here the $\pm$-superscripts stand for the sign of λ and $\rho^{(\alpha,\,\beta)} = (1-x)^\alpha(1+x)^\beta$ is the weight function for Jacobi polynomials $P^{(\alpha,\,\beta)}(x)$ defined in the Appendix A. The overall multiplicative constants in these expressions were chosen in order to fix the right signs of matrix elements (45) and ensure the correct behavior of spherical functions under complex conjugation (47).

The relation between Υ^+ and Υ^- is very simple:

$$\Upsilon^\pm_{lm}(x,\,\phi) = \pm i\sigma_z\,\Upsilon^\mp_{lm}(x,\,\phi). \tag{39}$$

The structure of this representation allows to prove directly the orthogonality of spherical functions:

$$\langle\Upsilon^{\varepsilon_1}_{l_1m_1}|\Upsilon^{\varepsilon_2}_{l_2m_2}\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} (\Upsilon^{\varepsilon_1}_{l_1m_1})^\dagger\Upsilon^{\varepsilon_2}_{l_2m_2}\,\sin\theta\,d\theta = \delta^{\varepsilon_1\varepsilon_2}\,\delta_{l_1l_2}\,\delta_{m_1m_2}. \tag{40}$$

Integration over $d\phi$ generates the Kronecker $\delta_{m_1 m_2}$-symbol. After that the integrand becomes a sum of two expressions of the type $\rho^{(|m^{\pm}|,\,|m^{\mp}|)} P_{l_1-|m|}^{(|m^{\pm}|,\,|m^{\mp}|)} P_{l_2-|m|}^{(|m^{\pm}|,\,|m^{\mp}|)}$ that adverts us to the orthogonality of Jacobi polynomials (A.5). If the only distinction between the Υ-functions is $\epsilon_1 \neq \epsilon_2$ one should turn back to eqn. (39). Due to the presence of σ_z for $\epsilon_1 = -\epsilon_2$ the equal contributions of upper and lower spinor components are subtracted from each other and cancel.

Two more economic representations can be found with the help of identities (A.9) of the Appendix. The first is:

$$\begin{aligned} \Upsilon^{\pm}_{lm}(x,\,\phi) &= \pm\frac{i^{l^+}(-1)^{l^-}}{2^{l^+}\Gamma(l^+)}\sqrt{\frac{(l+m)!}{(l-m)!}} \\ &\times \frac{e^{im\phi}}{\sqrt{2\pi}}\begin{pmatrix} \sqrt{\mp i}(1-x)^{-\frac{m^-}{2}}(1+x)^{-\frac{m^+}{2}}\frac{d^{l-m}}{dx^{l-m}}(1-x)^{l^-}(1+x)^{l^+} \\ \sqrt{\pm i}(1-x)^{-\frac{m^+}{2}}(1+x)^{-\frac{m^-}{2}}\frac{d^{l-m}}{dx^{l-m}}(1-x)^{l^+}(1+x)^{l^-} \end{pmatrix}. \end{aligned} \tag{41}$$

The second one differs from it mainly by the sign of m.

$$\begin{aligned} \Upsilon^{\pm}_{lm}(x,\,\phi) &= \frac{i^{l^+}(-1)^{l-m}}{2^{l^+}\Gamma(l^+)}\sqrt{\frac{(l-m)!}{(l+m)!}} \\ &\times \frac{e^{im\phi}}{\sqrt{2\pi}}\begin{pmatrix} \mp\sqrt{\mp i}(1-x)^{\frac{m^-}{2}}(1+x)^{\frac{m^+}{2}}\frac{d^{l+m}}{dx^{l+m}}(1-x)^{l^+}(1+x)^{l^-} \\ \pm\sqrt{\pm i}(1-x)^{\frac{m^+}{2}}(1+x)^{\frac{m^-}{2}}\frac{d^{l+m}}{dx^{l+m}}(1-x)^{l^-}(1+x)^{l^+} \end{pmatrix}. \end{aligned} \tag{42}$$

The important feature of these functions is that they are not zero only for values of m ranging from $-l$ to l. Hence each multiplet contains $2l+1$ terms. Indeed once $l = n+|m|$ is a half-integer then l^+ and l^- in their turn must be integers. As a result the products $(1-x)^{l^{\pm}}(1+x)^{l^{\mp}}$ are polynomials of order $2l$ and their derivatives of orders higher than $2l$ are zero.

These expressions are perfectly suited for acting on them by $\hat{L}_+$ and $\hat{L}_-$ operators. First let us note that for any m

$$\hat{L}_-\, e^{im\phi}\begin{pmatrix} \left[\rho^{(m^-,\,m^+)}\right]^{-\frac{1}{2}} \\ \left[\rho^{(m^+,\,m^-)}\right]^{-\frac{1}{2}} \end{pmatrix} = \hat{L}_+\, e^{im\phi}\begin{pmatrix} \left[\rho^{(m^-,\,m^+)}\right]^{\frac{1}{2}} \\ \left[\rho^{(m^+,\,m^-)}\right]^{\frac{1}{2}} \end{pmatrix} = 0. \tag{43}$$

If we take Υ in the form (41) when acting by $\hat{L}_-$ and in the form (42) when acting by $\hat{L}_+$ we shall readily find that

$$\hat{L}_-\,\Upsilon_{l,\,m} = \sqrt{(l+m)(l-m+1)}\,\Upsilon_{l,\,m-1}; \tag{44a}$$

$$\hat{L}_+\,\Upsilon_{l,\,m} = \sqrt{(l+m+1)(l-m)}\,\Upsilon_{l,\,m+1}. \tag{44b}$$

Thus we retain the standard expressions for the matrix elements of momentum operators,

$$\langle l,\,m-1\,|\hat{L}_-|\,l,\,m\rangle = \langle l,\,m\,|\hat{L}_+|\,l,\,m-1\rangle = \sqrt{(l+m)(l-m+1)}; \tag{45a}$$

$$\langle l,\,m\,|\hat{L}_z|\,l,\,m\rangle = m. \tag{45b}$$

Deducing these relations directly from formulae (38) would require engaging but to (43) the Jacobi polynomial differentiation formula (A.6).

2.4 Complex conjugation and time-reversal

Finally I would like to comment the choice of coefficients in definition (38). We have already mentioned that insofar as the momentum operators (31) are diagonal the upper and lower spinor components of $\Upsilon_{l,m}$ belong to different representations of the $SU(2)$-group. The eigenvalue equation (16) and normalization condition (29) link the components to each other and fix their ratio (28) and absolute values. In the mean time the overall complex phase remains free. It may be determined from the behaviour of eigenfunctions under the complex conjugation. The latter in its own turn is closely related to properties with respect to time-reversal symmetry in $2+1$-dimensions. Action of the time-reversal (T) transformation onto spinors is described by the formulae [15]:

$$T : \psi \rightarrow -i\sigma_y\, \psi^*. \tag{46a}$$

The $-i\sigma_y$ matrix guarantees that the time reversed spinors belong to the same fundamental representation of $SU(2)$ group and not to the complex-conjugated one. This definition agrees with the time-reversal properties of ordinary spherical functions $Y_{l,m}$ (see Appendix B)

$$T : Y_{l,m} \rightarrow Y^*_{l,-m} = (-1)^{l-m} Y_{l,-m}. \tag{46b}$$

Definitions (46) guarantee the equivalence of integer-momentum representations of $SU(2)$ written both in terms of spinors and spherical functions.

The same line was pursued by our definitions of Υ-functions. The idea was that the T-transformation must convert Υ's into themselves: $T : \Upsilon_{l,m} \rightarrow (-1)^{l-m}\,\Upsilon_{l,-m}$. This relates components of the complex conjugated spinor $\Upsilon^*_{l,m}$ to those of $\Upsilon_{l,-m}$ as follows:

$$T : \Upsilon_{l,m} = -i\sigma_y \Upsilon^*_{l,m} = \begin{pmatrix} -\beta^*_{l,m} \\ \alpha^*_{l,m} \end{pmatrix} = (-1)^{l-m} \begin{pmatrix} \alpha_{l,-m} \\ \beta_{l,-m} \end{pmatrix}. \tag{47}$$

One may show that for the functions introduced in the previous section this is indeed the case. The simplest way to is to take $\Upsilon_{l,m}$ and $\Upsilon_{l,-m}$ in different representations, say (41) and (42). A direct comparison of the coefficients proves (47) (mark that $(-m)^{\pm} = -(m^{\mp})$).

The time-reversal transformation performed twice changes the sign of the eigenfunctions:

$$T^2 : \Upsilon_{l,m} = (-1)^{2l} \Upsilon_{l,m} = -\Upsilon_{l,m}. \tag{48}$$

This also complies with what should be expected from general considerations for representations with half-integer momentum.

3 Relation between Υ and Ω spherical spinors

3.1 Cartesian coordinates

Now we shall study the relation of the newly found functions Υ to the standard spherical spinors. In order to do this we are going to convert Υ-functions to Cartesian coordinates. This requires passing from the two-dimensional Riemann sphere S^2 to the three-dimensional Euclidean space. In order to access the new dimension we must first add to the two polar angles $q^1 = \theta$ and $q^2 = \phi$ the radial coordinate $q^3 = r$. The relations between the 3-dimensional spherical and Euclidean coordinates are standard:

$$x^1 = x = r \sin\theta \cos\phi; \quad x^2 = y = r \sin\theta \sin\phi; \quad x^3 = z = r \cos\theta. \tag{49}$$

The old metric tensor and zweibein on the sphere (2) must be substituted by the new spherical metric and dreibein

$$g_{\alpha\beta} = \mathrm{diag}(r^2,\, r^2 \sin^2\theta,\, 1); \qquad e^a_\alpha = \mathrm{diag}(r,\, r\sin\theta,\, 1). \tag{50}$$

The place of the third Dirac matrix is taken by $\gamma^3 = \sigma_z$.

3.2 Spinor transformation rules

Now we would like to transform our formulae to the Cartesian coordinates. Note that the transformation includes a rotation of the dreibein. Accordingly the spinors will change. Let us remind the corresponding transformation rules.

Local rotations of the dreibein associated with coordinate changes generate the spin connection. The spinor and dreibein connections depend on each other so that γ^a-matrices with dreibein indices stay constant.

In order to find the explicit form of spinor transformation let us consider an auxiliary local bilinear $A^\alpha(q) = \psi^\dagger(q)\sigma^\alpha\psi(q)$ with $\psi(q)$ being an arbitrary spinor. Let q^α be the old and x^μ the new coordinates (there was no reason yet to claim that x^m were Euclidean). Being a simple contravariant vector A obeys the following transformation law:

$$A^\mu(x) = \frac{\partial x^\mu}{\partial q^\alpha} A^\alpha(q) \quad \text{and} \quad A^a = e^a_\mu(x) \frac{\partial x^\mu}{\partial q^\alpha} e^\alpha_b(q)\, \psi^\dagger(q)\sigma^b\psi(q), \tag{51}$$

where x and q refer to the same spatial point. Take the note of the two different dreibeins that appear in this expression: the first one $e^a_\alpha(q)$ refers to the original frame and $e^b_\mu(x)$ to the new one. The matrix $U^a_b = e^a_\mu(x) \frac{\partial x^\mu}{\partial q^\alpha} e^\alpha_b(q)$ is orthogonal $\tilde{U}\, U = U\, \tilde{U} = 1$ (where $\tilde{U}$ denotes the transposed matrix $\tilde{U}^a_b = U^b_a$). It describes the local rotation of objects with dreibein indices that arise from the coordinate transformation. Let us introduce into equation (51) an auxiliary unitary 2×2 matrix $V(q)$ $(V\, V^\dagger = 1)$:

$$A^a = U^a_b\, \psi^\dagger \sigma^\alpha \psi = e^a_\mu(x) \frac{\partial x^\mu}{\partial q^\alpha} e^\alpha_b(q)\, \psi^\dagger(q)\, V V^\dagger\, \sigma^b\, V V^\dagger\, \psi(q). \tag{52}$$

We shall show shortly that it is always possible to choose V so that it will compensate the rotation U of the dreibein putting the Pauli matrices back into the conventional form:

$$e^a_\mu(x) \frac{\partial x^\mu}{\partial q^\alpha} e^\alpha_b(q)\, V^\dagger(q)\, \sigma^b\, V(q) = \sigma^a. \tag{53}$$

Provided that this condition holds the σ-matrices will stay the same in the new coordinate frame. The associated transformation rule for spinors must be:

$$\psi(x) = V^\dagger\, \psi(q), \qquad \psi^\dagger(x) = \psi^\dagger(q)\, V. \tag{54}$$

Let us calculate the V-matrix. Similarly to any other rotation the matrix U^a_b may be written in terms of the adjoint $O(3)$ rotation generators $\hat{L}$ as:

$$\left[\hat{L}^a\right]^b_c = -i\, \epsilon^{abc}. \tag{55}$$

Let unit vector $n_c(q)$ point along the axis and $\theta(q)$ be the angle of local rotation. Then

$$U^a_b(q) = \left[\exp i\hat{L}^c\, n_c(q)\theta(q)\right]^a_b = n^a n_b + (\delta^a_b - n^a n_b)\cos\theta + i\, n_c \left[\hat{L}^c\right]^a_b \sin\theta. \tag{56}$$

From the commutation relations of the Pauli matrices it is straightforward to see that

$$\left[\exp i\hat{L}^c\, n_c\theta\right]^a_b \left(\exp\frac{i}{2}\sigma^c n_c\theta\, \sigma^b \exp -\frac{i}{2}\sigma^c\, n_c\theta\right) = \sigma^a. \tag{57}$$

Therefore the matrices V and $V^\dagger$ compensating the rotation (56) are

$$V(q) = \exp -\frac{i}{2}\sigma^c n_c(q)\theta(q) \qquad \text{and} \qquad V^\dagger(q) = \exp\frac{i}{2}\sigma^c n_c(q)\theta(q). \tag{58}$$

Thus the transformation of spinors is unambiguously determined by the rotation angle θ and direction of the axis n which may be found either from geometrical considerations or algebraically by solving equation (53).

3.3 Cartesian realization of Υ-functions

Now we can apply the theory and find out what do Υ-spinors look like in the ordinary Cartesian coordinates. First we shall derive the general formulae transforming spinors from the spherical to the orthogonal frame and then apply them to the spinors in question. Taking for input the relation between spherical and orthogonal coordinates (49), the explicit spherical dreibein (50) and the trivial Cartesian dreibein $e^a_\mu = \delta^a_\mu$ we obtain the rotation matrix ($e^a_\mu = \delta^a_\mu$).

$$U^a_b = e^a_\mu(x)\,\frac{\partial x^\mu}{\partial q^\alpha}\, e^\alpha_b(q) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{59}$$

This may be decomposed into the product of two successive turns:

$$U^a_b(q) = \left[\exp -i\,\phi\hat{L}_3 \exp -i\theta\hat{L}_2\right]^a_b. \tag{60}$$

Geometrically these two rotations align the local axes of spherical frame with those of Cartesian one. Obviously θ and ϕ are nothing but the corresponding Euler angles. Spinors must also suffer the same two successive rotations with V-matrices being:

$$V = \exp\frac{i\sigma_y}{2}\theta\, \exp\frac{i\sigma_z}{2}\phi \;=\; \begin{pmatrix} e^{\frac{i\phi}{2}}\cos\frac{\theta}{2} & e^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ -e^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & e^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}; \tag{61a}$$

$$V^\dagger = \exp -\frac{i\sigma_z}{2}\phi\, \exp -\frac{i\sigma_y}{2}\theta \;=\; \begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -e^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ e^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & e^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}. \tag{61b}$$

Now we shall perform the V-rotation on spinor spherical functions Υ (the trick is to use representation (41) when calculating the upper line and apply (42) to the lower one):

$$V^\dagger\,\Upsilon^\pm_{l\,m} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{l+m}{2l}}\,Y_{l^-m^-} \pm \sqrt{\frac{l-m+1}{2l+2}}\,Y_{l^+m^-} \\ \sqrt{\frac{l-m}{2l}}\,Y_{l^-m^+} \mp \sqrt{\frac{l+m+1}{2l+2}}\,Y_{l^+m^+} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\Omega_{l,l^-,m} \mp \Omega_{l,l^+,m}\right). \tag{62}$$

Here $Y_{l\,m}$ are spherical functions (see Appendix B) and $\Omega_{j,\,l,\,m}$ are the ordinary spherical spinors that realize an alternative representation of spin-$\frac{1}{2}$ states on the sphere [11]. The quantum numbers characteristic of a spinor $\Omega_{j,\,l,\,m}$ are: the half-integer total angular momentum $j > 0$, integer orbital momentum $l = j \pm \frac{1}{2}$ and projection of the total momentum onto the polar axis m (that is again a half-integer). They are the two-component functions defined by the formulae

$$\Omega_{l+\frac{1}{2},\,l,\,m} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}\,Y_{l,\,m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}}\,Y_{l,\,m+\frac{1}{2}} \end{pmatrix} \qquad \text{and} \qquad \Omega_{l-\frac{1}{2},\,l,\,m} = \begin{pmatrix} -\sqrt{\frac{j-m+1}{2j+2}}\,Y_{l,\,m-\frac{1}{2}} \\ \sqrt{\frac{j+m+1}{2j+2}}\,Y_{l,\,m+\frac{1}{2}} \end{pmatrix}. \tag{63}$$

Similarly to Υ the Ω-spinors make an orthonormal basis in the space of spinor functions on S^2:

$$\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \Omega_{j\,l\,m}(\cos\theta,\,\phi)\, \Omega_{j'\,l'\,m'}(\cos\theta,\,\phi) = \delta_{j\,j'}\,\delta_{l\,l'}\,\delta_{m\,m'}. \tag{64}$$

We conclude that in Cartesian coordinates the functions Υ may be expressed in terms of the ordinary spherical spinors Ω (see (63)) but do not coincide with those. Representation (62) helps finding the standard quantum numbers of Υ-spinors. Inasmuch as they are superpositions of states with equal values of j and m Υ-functions must the same values of total angular momentum $\hat{J}$ and its z-projection $\hat{J}_z$:

$$\hat{\mathbf{J}}^2\,\Upsilon_{lm} = (\hat{\mathbf{L}}_C + \hat{\mathbf{S}}_C)^2\,\Upsilon_{lm} = l(l+1)\,\Upsilon_{lm} \qquad \text{and} \qquad \hat{J}_z\,\Upsilon_{lm} = m\,\Upsilon_{lm}; \tag{65}$$

where $\hat{\mathbf{L}}_C$ and $\hat{\mathbf{S}}_C$ are the Cartesian operators of vectors of orbital momentum and spin respectively. In the mean time being composed of states with different orbital momenta l^+ and l^- our spinors do not diagonalize $\hat{\mathbf{L}}^2_C$. Still it is interesting to calculate its average:

$$\langle\Upsilon_{lm}|\hat{\mathbf{L}}^2_C|\Upsilon_{lm} >= \frac{1}{2}\left[l^+(l^++1) + l^-(l^-+1)\right] = \left(l+\frac{1}{2}\right)^2 = \lambda^2. \tag{66}$$

You see that the average value of $\hat{\mathbf{L}}^2_C$ for spherical spinors Υ is equal to the squared eigenvalue of spherical Dirac operator. Clearly this is a mere coincidence since in this representation $-i\hat{\nabla}$ is diagonal whereas $\hat{\mathbf{L}}^2_C$ is not.

Now it is the time to compare the two types of spinors. Either of them forms a complete orthogonal functional system and belongs to the same representation of the $SU(2)$ group. Moreover Υ's are linear combinations of Ω's and *vice versa.* The question arises what is the origin of distinction between them.

The answer is very simple. Both Ω and Υ diagonalize one more operator but to $\hat{J}$ and $\hat{J}_z$. These are the square of orbital momentum $\hat{\mathbf{L}}^2_C$ for Ω and spherical Dirac operator for Υ. The different choice of supplementary operators leads to different representations.

The ordinary spinors Ω were obtained from solving in the flat space the three-dimensional Laplace equation on spinors

$$-\Delta\,\Psi = \left[\frac{1}{r^2}\hat{\mathbf{L}}^2_C + \frac{1}{r^2}\,\partial_r\,r^2\,\partial_r\right]\,\Psi = k^2\,\Psi. \tag{67}$$

The Laplace operator is diagonal with respect to spin so that the upper and lower components of Ψ behave as independent scalars. Separating the radial and angular dependencies one arrives at solutions with integer values of orbital momentum $l_C = 0,\,1,\ldots$ Then those are united into doublets with definite total momentum j and its projection $j_z = m$.

Now the Υ-spinors are found by diagonalizing Dirac operator on the Riemann sphere that is an essentially curved manifold. One may clearly mark the difference from flat Euclidean space from the curvature term that manifestly enters the square of Dirac operator (14). Besides the nontrivial spin connection makes even the covariant two-dimensional Laplace operator (15) on the sphere differ from the angular part of (67).

To summarize one may say that as a result Ω-spinors are better suited for separation of variables in spherically symmetric problems of non-relativistic quantum mechanics. On the other hand Υ-functions may be useful when one is specially interested in properties of fermions localized on the sphere.

In conclusion we present for reference the Cartesian realization of Dirac operator (11) $-i\hat{\nabla}_C = -i\, V^{\dagger}\hat{\nabla}V$ with V-matrices given by (59)

$$\begin{aligned} -i\hat{\nabla}_C &= -i\sigma_x\, e^{i\phi\sigma_z}\,(\cos\theta\,\partial_\theta - \sin\theta) + i\sigma_z\,(\sin\theta\,\partial_\theta + \cos\theta) - \frac{i\sigma_y}{\sin\theta}\, e^{i\phi\sigma_z}\,\partial_\phi \\ &= -ie^{-\frac{i\sigma_z\phi}{2}}\,\sigma_x\,\partial_\theta\, e^{i\sigma_y\theta}\, e^{\frac{i\sigma_z\phi}{2}} - \frac{i\sigma_y}{\sin\theta}\, e^{i\phi\sigma_z}\,\partial_\phi. \end{aligned} \tag{68}$$

Calculation of $(-i\hat{\nabla}_C)^2$ is left to the reader as an exercise.

Conclusion

We have found the eigenfunctions of Dirac operator on the Riemann sphere S^2. This family of orthonormal two-component spinors may be used for basis in problems involving fermions in spherically symmetric fields. The eigensponors are expressed in terms of Jacobi polynomials. The solutions depend on spherical angles and are characterized by the following quantum numbers:

- eigenvalues of Dirac operator λ which are nonzero either positive or negative integers ($\lambda = 0$ is excluded by Lichnerovicz theorem);
- half-integer total angular momentum $l = |\lambda| - \frac{1}{2}$;
- projection of angular momentum onto the polar axis $m = -l, \ldots l$ which is obviously a half-integer.

The set of solutions corresponding to highest weight l realizes a $2l + 1$ dimensional representation of $SU(2)$ group with matrix elements being given by the usual formulae for half-integer spin. The normalization coefficients are chosen so that the properties of our solutions with respect to complex conjugation and time reversal are in accord with those of spherical functions.

The eigenspinors of spherical Dirac operator Υ differ from the ordinary spherical spinors Ω. These two types of spinors diagonalize the different sets of operators. Therefore Ω-spinors are listed by another set of quantum numbers. It includes: half-integer total angular momentum j; integer orbital momentum $l = j \pm \frac{1}{2}$; half-integer projection m of angular momentum onto the polar axis.

The second source of difference is the curvature of S^2. The functions Υ diagonalize the Dirac operator on the Riemann sphere which is a genuinely curved manifold. Therefore the spin connection is inevitably present in the equation. On the other hand the conventional spherical spinors are found by separating the angular variables in the flat three-dimensional space R^3. Obviously the resulting equations contain no spin connection unless one inserts it by hand.

As a consequence the Υ spherical spinors must be better suited for problems formulated on the Riemann sphere (see Introduction for examples). We hope that the explicit and detailed presentation of properties of Υ-spinors will help to use them for practical needs.

In conclusion I would like to thank Maximillian Kreuzer for the kind hospitality and discussions during the preparation of the pilot version of the paper. I'm particularly grateful to Joseph C. Varilly for introducing me to the history of the subject and rendering me excerpts from his own work.

The work was done with partial support from RFBR grants 00–02–17808 and 00–15–96786.

A Properties of Jacobi polynomials

Jacobi polynomials $P_n^{(\alpha,\,\beta)}(x)$ ($\alpha,\,\beta > -1$, n being the order) are classical orthogonal polynomials that satisfy the equation of hypergeometric type [13, 14]

$$\sigma(x)\,y'' + \tau(x)\,y' + \lambda_n\,y = \frac{1}{\rho(x)}\frac{d}{dx}[\sigma(x)\,\rho(x)\,y'] + \lambda_n\,y = 0, \tag{A.1}$$

with the coefficient functions:

$$\rho^{(\alpha,\,\beta)}(x) = (1-x)^\alpha(1+x)^\beta; \qquad \sigma(x) = 1 - x^2; \qquad \tau(x) = \beta - \alpha - (\alpha+\beta+2)x. \tag{A.2}$$

The constant λ_n is equal to

$$\lambda_n = n(n+\alpha+\beta+1). \tag{A.3}$$

The explicit form of Jacobi polynomials is given by the differential and integral Rodrigues' formulas:

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(-1)^n}{2^n\,n!}\frac{1}{\rho(x)}\frac{d^n}{dx^n}\left[\sigma^n(x)\,\rho(x)\right]; \tag{A.4a}$$

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(-1)^n}{2^n\,n!}\frac{1}{\rho(x)}\frac{n!}{2\pi i}\oint\frac{\sigma^n(z)\,\rho(z)}{(z-x)^{n+1}}\,dz. \tag{A.4b}$$

The contour of complex integration in the second equation must encircle the point x.

Jacobi polynomials with given α and β are orthogonal on the interval $[-1,\,1]$ with the weight $\rho^{(\alpha,\,\beta)}(x)$

$$\int_{-1}^{1} P_m^{(\alpha,\,\beta)}(x) P_n^{(\alpha,\,\beta)}(x)\,\rho^{(\alpha,\,\beta)}(x)\,dx = \delta_{mn}\,\frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)}. \tag{A.5}$$

Jacobi polynomials form on this interval a complete set and any square integrable function can be expanded in terms of $P_n^{(\alpha,\,\beta)}$.

Adjacent polynomials are related to each other by the differentiation formula:

$$\frac{d}{dx}P_n^{(\alpha,\,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)\,P_{n-1}^{(\alpha+1,\,\beta+1)}(x). \tag{A.6}$$

In the paper we met Jacobi polynomials with $\alpha = \beta \pm 1$ and used their special properties. The first two identities help to relate the upper and lower components of spinor spherical functions (25). Let us prove that for $k \geq 1$

$$\left(\frac{k}{1-x} - \frac{d}{dx}\right) P_n^{(k,\,k-1)}(x) = \frac{n+k}{1-x}\, P_n^{(k-1,\,k)}(x); \qquad \text{(A.7a)}$$

$$\left(\frac{k}{1-x} + \frac{d}{dx}\right) P_n^{(k-1,\,k)}(x) = \frac{n+k}{1+x}\, P_n^{(k,\,k-1)}(x). \qquad \text{(A.7b)}$$

The proof is based on the integral Rodrigues' formula (A.4b). Let $C = \frac{1}{2\pi i}\left(-\frac{1}{2}\right)^n$ be the numerical factor common to the both sides of the equations. Then after differentiation we get:

$$\left(\frac{k}{1-x} - \frac{d}{dx}\right) P_n^{(k,\,k-1)} = \frac{n+k}{1+x} P_n^{(k,\,k-1)} - \frac{C(n+1)}{(1-x^2)^k} \oint \frac{(1-z^2)^{n+k}\,dz}{(z-x)^{n+2}}; \qquad \text{(A.8a)}$$

$$\left(\frac{k}{1+x} + \frac{d}{dx}\right) P_n^{(k-1,\,k)} = \frac{n+k}{1-x} P_n^{(k-1,\,k)} + \frac{C(n+1)}{(1-x^2)^k} \oint \frac{(1-z^2)^{n+k}\,dz}{(z-x)^{n+2}}. \qquad \text{(A.8b)}$$

Integrating the rightmost terms by parts opens the way to the desired result (A.7).

The second pair of relations was used in deducing representations (41, 42). They are valid only for integer α and β and for convenience we shall use the notation of Section 2.3. (We remind that both $l^{\pm} = l \pm \frac{1}{2} \geq 0$ and $m^{\pm} = m \pm \frac{1}{2}$ are integers.)

$$\frac{d^{l+m}}{dx^{l+m}}(1-x)^{l^-}(1+x)^{l^+} = (-1)^{m^-}\frac{(l+m)!}{(l-m)!}\frac{\frac{d^{l-m}}{dx^{l-m}}(1-x)^{l^+}(1+x)^{l^-}}{(1-x)^{m^+}(1+x)^{m^-}}; \qquad \text{(A.9a)}$$

$$\frac{d^{l+m}}{dx^{l+m}}(1-x)^{l^+}(1+x)^{l^-} = (-1)^{m^+}\frac{(l+m)!}{(l-m)!}\frac{\frac{d^{l-m}}{dx^{l-m}}(1-x)^{l^-}(1+x)^{l^+}}{(1-x)^{m^-}(1+x)^{m^+}}. \qquad \text{(A.9b)}$$

For example let us prove (A.9a) for $m > 0$. Note that relation (A.9b) extends its validity to negative m and *v. v.* The $l+m$-th derivative of the product is

$$A = \frac{d^{l+m}}{dx^{l+m}}(1-x)^{l^-}(1+x)^{l^+} = \sum_{k=0}^{l+m}\binom{l+m}{k}\frac{d^k}{dx^k}(1-x)^{l^-}\,\frac{d^{l+m-k}}{dx^{l+m-k}}(1+x)^{l^+}. \qquad \text{(A.10)}$$

Insofar as l^+, l^- are integers the derivatives of $(1-x)^{l^-}$ and $(1+x)^{l^+}$ of orders higher than l^- and l^+ vanish. Thus only the terms with $m^- \leq k \leq l^-$ survive. Introducing a new summation index $j = k - m^-$ we obtain after expanding the derivatives

$$A = \sum_{j=0}^{l-m}\frac{(-1)^{j+m^-}(l+m)!}{(m^-+j)!(l^+-j)!}\,\frac{l^-!(1-x)^{l-m-j}}{(l-m-j)!}\,\frac{l^+!(1+x)^j}{j!}. \qquad \text{(A.11)}$$

Observing that

$$\frac{(-1)^j l^+!}{(l^+-j)!}(1-x)^{l-m-j} = \frac{\frac{d^j}{dx^j}(1-x)^{l^+}}{(1-x)^{m^+}} \quad \text{and} \quad \frac{l^-!}{(m^-+j)!}(1+x)^j = \frac{\frac{d^{l-m-j}}{dx^{l-m-j}}(1+x)^{l^-}}{(1+x)^{m^-}}, \qquad \text{(A.12)}$$

we immediately find

$$A = \frac{(l+m)!}{(l-m)!}\frac{(-1)^{m^-}}{(1-x)^{m^+}(1+x)^{m^-}}\sum_{j=0}^{l-m}\binom{l-m}{j}\frac{d^j}{dx^j}(1-x)^{l^+}\,\frac{d^{l-m-j}}{dx^{l-m-j}}(1+x)^{l^-}. \qquad \text{(A.13)}$$

This is nothing but the RHS of equation (A.9a). The proof of (A.9b) may be carried out by simply reversing the sign $x \to -x$.

B Spherical functions

Here we cite for reference the basics of spherical functions and Legendre polynomials.

The associated Legendre polynomials $P_l^m(x)$ with l, m being nonnegative integers are solutions to the second order ordinary differential equation [13, 14]

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}\,y - \frac{m^2}{1-x^2}\,y + l(l+1)\,y = 0. \tag{B.1}$$

The two convenient representations of functions $P_l^m(x)$ are:

$$P_l^m(x) = \frac{(-1)^l}{2^l\,l!}\,(1-x^2)^{\frac{m}{2}}\,\frac{d^{l+m}}{dx^{l+m}}(1-x^2)^l \tag{B.2a}$$

$$P_l^m(x) = \frac{(-1)^{l-m}}{2^l\,l!}\,\frac{(l+m)!}{(l-m)!}\,(1-x^2)^{-\frac{m}{2}}\,\frac{d^{l-m}}{dx^{l-m}}(1-x^2)^l. \tag{B.2b}$$

They may be also expressed in terms of the Jacobi and Legendre polynomials,

$$P_l^m(x) = \frac{(l+m)!}{2^m\,l!}\,(1-x^2)^{\frac{m}{2}}\,P_{l-m}^{(m,m)}(x), \qquad \text{and} \qquad P_l^m(x) = (1-x^2)^{\frac{m}{2}}\,\frac{d^m}{dx^m}P_l(x). \tag{B.3}$$

From the orthogonality of Jacobi polynomials (A.5) it follows that associated Legendre polynomials with the same value of m are orthogonal on the interval $[-1, 1]$ with the unit weight:

$$\int_{-1}^{1} dx\,P_k^m(x)\,P_l^m(x) = \delta_{kl}\,\frac{(l+m)!}{(l-m)!}\,\frac{2}{2l+1}. \tag{B.4}$$

Spherical functions $Y_{l\,m}(x,\,\phi)$ are listed by two integers, namely, by the value of the angular momentum $l \geq 0$ and its projection onto the polar axis $-l \leq m \leq l$. The functions $Y_{l\,m}(x,\,\phi)$ are normalized eigenfunctions of the scalar (spin-zero) Laplace operator on the sphere:

$$\left(-\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right)\,Y_{l\,m}(\cos\theta,\,\phi) = l(l+1)\,Y_{l\,m}(\cos\theta,\,\phi). \tag{B.5}$$

They may be expressed in terms of the associated Legendre polynomials:

$$Y_{l\,m}(x,\,\phi) = (-1)^{\frac{m+|m|}{2}}\,i^l\,\frac{e^{im\phi}}{2\pi}\,\sqrt{\frac{2l+1}{2}\,\frac{(l-|m|)!}{(l+|m|)!}}\,P_l^{|m|}(x). \tag{B.6}$$

With the help of identities (B.2) the spherical functions may be written as

$$Y_{l\,m}(x,\,\phi) = \frac{(-1)^l\,i^l}{2^l\,l!}\,\frac{e^{im\phi}}{2\pi}\,\sqrt{\frac{2l+1}{2}\,\frac{(l+m)!}{(l-m)!}}\,(1-x^2)^{-\frac{m}{2}}\,\frac{d^{l-m}}{dx^{l-m}}(1-x^2)^l. \tag{B.7a}$$

$$Y_{l\,m}(x,\,\phi) = \frac{(-1)^{l-m}\,i^l}{2^l\,l!}\,\frac{e^{im\phi}}{2\pi}\,\sqrt{\frac{2l+1}{2}\,\frac{(l-m)!}{(l+m)!}}\,(1-x^2)^{\frac{m}{2}}\,\frac{d^{l+m}}{dx^{l+m}}(1-x^2)^l. \tag{B.7b}$$

The functions $Y_{lm}(\cos\theta,\,\phi)$ form on the sphere the orthonormal functional basis:

$$\int_0^{\pi} \sin\theta\,d\theta\,\int_0^{2\pi} d\phi\,Y_{l\,m}(\cos\theta,\,\phi)\,Y_{l'\,m'}(\cos\theta,\,\phi) = \delta_{l\,l'}\,\delta_{m\,m'}. \tag{B.8}$$

Inasmuch as the angular part of 3-dimensional Laplace operator (B.5) coincides with the square of angular momentum $\hat{\mathbf{L}}_C^2$ in coordinate representation spherical functions $Y_{l\,m}$ are the basis vectors of $2l+1$ dimensional representation of rotation group $O(3)$.

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